Last Time

• States
• Finite State Machines (DFAs)
  – states, alphabet, transition function, start state, accept state(s)
• Language Function
One More Example
One More Example

- states: \( Q = \{ q_1, q_2 \} \)
- alphabet: \( \Sigma = \{ 0, 1 \} \)
- transition function: \( \delta: Q \times \Sigma \rightarrow Q \)

\[
\begin{array}{c|cc}
& 0 & 1 \\
q_1 & q_1 & q_2 \\
q_2 & q_1 & q_2 \\
\end{array}
\]

- start state: \( q_1 \)
- accept states: \( F = \{ q_1 \} \)
What Strings Will This Accept?

- Accepts $\varepsilon$
- Rejected 1
- Rejected 00
- Rejected 000

$L(M) = \{w \mid w \text{ ends in a } 0 \text{ or is the empty string}\}$
What is this \( w \)?

- **Formal Definition of Computation:**
  - Let \( M = (Q, \Sigma, \delta, q_0, F) \) be a finite state machine, and let \( w = w_1 w_2 \ldots w_n \) be a string where each \( w_i \) is a member of the alphabet \( \Sigma \).
  - \( M \) accepts \( w \) if a sequence of states \( r_1, r_2, \ldots, r_n \) in \( Q \) exists with the following three conditions:
    1. \( r_0 = q_0 \)
    2. \( \delta(r_i, w_{i+1}) = r_{i+1} \) for \( i = 0, \ldots, n-1 \)
    3. \( r_n \in F \)

**IMPORTANT NOTE:**
We don’t know all of \( w \). We only know the next character.
Regular Languages

• A regular language is any language that is recognizable by a finite state machine.
  – (If we can build a DFA for it, it’s a regular language.)

• $M$ recognizes language $A$ if $A = \{ w \mid M \text{ accepts } w \}$. 
Operations on Regular Languages

• Let $A$ and $B$ be regular languages. We define the following allowable operations:
  
  **Union:** $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
  
  **Concatenation:** $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$
  
  **Star:** $A^* = \{x_1x_2 \ldots x_k \mid k \geq 0 \text{ and each } x_i \in A\}$
Toy Example

• Let the alphabet $\Sigma$ be the standard 26 letters \{a, b, ..., z\}. If $A = \{\text{good, bad}\}$ and $B = \{\text{boy, girl}\}$, then:

  – **Union**: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
    • $A \cup B = \{\text{good, bad, boy, girl}\}$
  
  – **Concatenation**: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$
    • $A \circ B = \{\text{goodboy, goodgirl, badboy, badgirl}\}$
  
  – **Star**: $A^* = \{x_1x_2 \ldots x_k \mid k \geq 0 \text{ and each } x_i \in A\}$
    • $A^* = \{\varepsilon, \text{good, bad, goodgood, goodbad, badgood, badbad, goodgoodgood, goodgoodbad, goodbadbad, ...}\}$
More Terminology

• **Unary Operation**: An operation that applies to only a single language. (e.g. star)

• **Binary Operation**: An operation that applies to two different languages. (e.g. union, concatenation)

• **Closed**: An operation is called “closed” if applying that operation to members of a set returns an object still in that set.
Are Regular Languages Closed Under the Union Operation?

• **Proof Idea:**
  – Take two regular languages; call them $A_1$ and $A_2$
  – Because $A_1$ and $A_2$ are regular, there exist machines that recognize them ($M_1$ and $M_2$)
  – Let’s try to create a new machine $M$ that simulates input $w$ on $M_1$ and $M_2$
  – $M$ should accept if either $M_1$ and $M_2$ accepts
Are Regular Languages Closed Under the Union Operation?

– **Problem:** We can’t “rewind” – once we read \( w \) on \( M_1 \), we can’t see it again for \( M_2 \)

– **Solution:** Keep track of both \( M_1 \) and \( M_2 \) simultaneously by remembering both states

  • If \( M_1 \) has \( k_1 \) states and \( M_2 \) has \( k_2 \) states, then \( M \) needs \( k_1 \times k_2 \) states!
Are Regular Languages Closed Under the Union Operation?

**Proof:** Let $M_1$ recognize $A_1$, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, and let $M_2$ recognize $A_2$, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.

Construct $M$ to recognize $A_1 \cup A_2$, where $M = (Q, \Sigma, \delta, q_0, F)$:

1. $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$
2. $\Sigma$ is the same as in $M_1$ and $M_2$
3. For each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, let
   $$\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$$
4. $q_0$ is the pair $(q_1, q_2)$
5. $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$. ■

What kind of proof was this? **Proof by Construction**
What about the Concatenation and Star Operations?

• We’ll get back to those later...
Any Questions?

HOMEWORK (due 9/9)

1.6dghim

Note: 1.6m asks for the empty set, not the empty string