CMPSC250
Lecture 11: Priority Queues and Heapsort

Prof. John Wenskovitch
02/15/2016
Last Time

• Quick Sort and Partitioning
  – Recursive algorithm like Merge Sort
  – Splits data based on a “pivot”
  – Another n*log(n) sorting algorithm
## Sorting Summary

<table>
<thead>
<tr>
<th>Sort</th>
<th>Compares</th>
<th>Exchanges/Array Accesses</th>
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</thead>
<tbody>
<tr>
<td>Selection Sort</td>
<td>$O(n^2)$</td>
<td>$O(n)$</td>
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<tr>
<td>Insertion Sort (Worst)</td>
<td>$O(n^2)$</td>
<td>$O(n^2)$</td>
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<tr>
<td>Insertion Sort (Average)</td>
<td>$O(n^2)$</td>
<td>$O(n^2)$</td>
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<tr>
<td>Insertion Sort (Best)</td>
<td>$O(n)$</td>
<td>0</td>
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<tr>
<td>Shell Sort (Worst)</td>
<td>$O(n^{3/2})$</td>
<td>$O(n^{3/2})$</td>
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<td>Shell Sort (Average)</td>
<td>It depends</td>
<td>It depends</td>
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<tr>
<td>Top-Down Mergesort (Worst)</td>
<td>$O(n \log(n))$</td>
<td>$O(n \log(n))$</td>
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<tr>
<td>Top-Down Mergesort (Average)</td>
<td>$O(n \log(n))$</td>
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<td>Bottom-Up Mergesort</td>
<td>$O(n \log(n))$</td>
<td>$O(n \log(n))$</td>
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<tr>
<td>Quicksort (Worst)</td>
<td>$O(n^2)$</td>
<td>$O(n^2)$</td>
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<tr>
<td>Quicksort (Average)</td>
<td>$O(n \log(n))$</td>
<td>$O(n \log(n))$</td>
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Online Data Acquisition

• Thus far, we have assumed that we have all of the data before we begin to sort.
  – What if we expect to continue to receive data throughout the sorting process? We’ll never get a totally sorted list.

• We also assumed that producing a fully sorted list was the end goal.
  – What if we don’t need everything sorted from smallest to biggest; what if we just want the smallest item in the array right now?
Priority Queue

• Two operations necessary:
  – Acquire new data (`insert()`)
  – Remove the largest item (`delMax()`) or the smallest item (`delMin()`).

• As we acquire new data, we assign it a **priority** (typically its value, but could be something calculated or defaulted).
  – Items with high/low priority will rise/fall in the data structure, so that they can be accessed quickly.
PQ Implementation #1: Unordered Array

- **insert()**: Simply add the new item to the end of the array.

- **delMax/Min()**: Find the largest (smallest) item in the array, swap it with what is at the end (single iteration of Selection Sort) and pop it off the end.

- **Insert cost**: $O(1)$

- **Remove cost**: $O(n)$
PQ Implementation #2: Ordered Array

• `insert()`: Add the new item to the end of the array, then push it to the correct location in the array (single iteration of Insertion Sort).

• `delMax/Min()`: Simply remove the item from the end of the list.

• **Insert cost**: $O(n)$
• **Remove cost**: $O(1)$
Heap

• Best of both worlds:
  – **Insert cost**: $O(\log(n))$
  – **Remove cost**: $O(\log(n))$

• A binary tree, where the key of each node is larger than or equal to the key of both of its children.
  – The largest key then will be located at the root.
Storing a Heap

- Represented sequentially in an array in **level order**, with the root at $a[1]$, its children at $a[2]$ and $a[3]$, ...
  - The children of node $k$ located at $2k$ and $2k+1$.
  - The height of a tree with $N$ items is $\lceil \log(N) \rceil$.
  - Moving items is a process called **reheapifying**.
Heap Functions – \texttt{swim()}

```java
while (k > 1 && less(k/2, k)) {
    exch(k/2, k);
    k = k/2;
} //while
```

(Also called \textbf{Bottom-Up Reheapify})

```java
private boolean less(int i, int j) {
    return pq[i].compareTo(pq[j]) < 0; }
```

```java
private void exch(int i, int j) {
    Key t = pq[i]; pq[i]=pq[j]; pq[j]=t; }
```
Heap Functions – `swim()`

The diagram illustrates a binary tree with nodes labeled `P`, `S`, `T`, `G`, `H`, `I`, `E`, `R`, `O`, and `A`. The tree shows the process of the `swim()` operation in a heap, which violates heap order by having a larger key than its parent.
Heap Functions – sink()

while (2*k <= N) {
    int j = 2*k;
    if (j < N && less(j, j+1)) {
        j++;
    } //if
    if (!less(k, j)) {
        break;
    } //if
    exch(k, j);
    k = j;
} //while

(Also called Top-Down Reheapify)
Heap Functions – \texttt{sink()}
Heaps and Priority Queues

• **insert()**: Add the new item to the end of the array, increment the size of the heap, and then **swim()** up through the heap to position the new item appropriately.

• **delMax()**: Remove the root of the heap, replace it with the item at the end of the array and decrement the size of the heap, then let that item **sink()** down through the heap to the appropriate position.
Building a Heap

1. Insert(P)  
2. Insert(Q)  
3. Insert(E)  
4. RemoveMax()  
5. Insert(X)  
6. Insert(A)  
7. Insert(M)  
8. RemoveMax()  
9. Insert(P)  
10. Insert(L)  
11. Insert(E)  
12. RemoveMax()
Heapsort

```java
int N = a.length
for (int k = N/2; k>=1; k--) {
    sink(a, k, N);
} //for

while (N > 1) {
    exch(a, 1, N--);
    sink(a, 1, N);
} //while
```
### Heapsort Visual

<table>
<thead>
<tr>
<th>N</th>
<th>k</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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**sorted result**

AEELMOPRSTX
Heapsort Evaluation

• **Theorem:** Sink-based heap construction uses fewer than 2N compares and fewer than N exchanges to construct a heap from N items.

• **Proof:** Most of the subheaps that we process are small. Consider a heap of 127 items:
  – We process 32 heaps of size 3, 16 heaps of size 7, 8 heaps of size 15, 4 heaps of size 31, 2 heaps of size 63, and 1 heap of size 127.
  – Multiply the number of heaps of each size by the height of each heap. \((32)(1) + (16)(2) + \ldots + (1)(6) = 120\) exchanges, and 240 compares.
Heapsort Evaluation

• **Theorem:** Heapsort uses fewer than $2n \times \log(n) + 2n$ compares (and half as many exchanges) to sort $n$ items.

• **Proof:** The $2n$ term covers the heap construction (previous slide). We run 1 sink operation for each of the $n$ items in the array, and the max cost of each sink is $\log(n)$.

• Heapsort grows at a rate of $O(n \times \log(n))$ with respect to both compares and exchanges.
Any Questions?