Last Time

• Greedy Algorithms
  – Take the best available solution at each step
  – Need to show correctness and optimality
• Proof by Exchange Argument
  – Minimum Spanning Tree problem
  – Minimum Average Waiting Time problem
• Other Problems
• Heuristics
Dynamic Programming

• **Dynamic Programming** – Solving a problem by breaking it down divide-and-conquer style into a set of smaller programs.
  – Only solve each subproblem once, no repeat computations.

• Basic approach:
  1. Design a top-down recursive algorithm.
  2. Convert it to a bottom-up iterative algorithm.
Toy Example: Fibonacci

- Top-down recursion:
  ```c
  int fibo(int val) {
    if (val == 0 || val == 1) {
      return 1;
    } else {
      return fibo(val-1) + fibo(val-2);
    } //if-else
  } //fibo
  ```
Toy Example: Fibonacci

• Bottom-up iterative:

```c
int fibo(int stop) {
    int val[0] = 0, val[1] = 1;
    for (int i = 2; i <= stop; i++) {
        val[i] = val[i-1] + val[i-2];
    } //for
    return val[stop];
} //fibo
```
Toy Example: Fibonacci

• So what did we do?
  – Start with a recursive algorithm that was $O(2^n)$. 
  – Pull out the base cases and initialize them with appropriate values.
  – Remove the recursion and turn it into a for loop, counting up from the base cases instead of counting down to them.
Subset Sum Problem

• **Input:**
  - Coin values \( v_1, v_2, \ldots, v_n \)
  - Total value \( L \)

• **Output:**
  - Change for \( L \) cents, if possible

• Example Input: 7, 2, 11, 9, 16, 13; \( L = 41 \)
• Example Output: \{11, 16, 14\} or \{2, 9, 16, 14\}
Subset Sum Problem

• For now, let’s not worry about the actual set of items. Instead, let’s see if a solution exists.

• Think about an approach like this:
  – First, is there a solution?
  – If so, generate that solution in a way that lets you recover old results.
  – Finally, post-process your computational results to find the exact solution.
Subset Sum Problem

• How do we compute if the first $j$ items in the set stay under the value $w$? (how can we build $\text{OPT}(j, w)$?)

• $\text{OPT}(0, w) = 0$

• $\text{OPT}(j, 0) = 1$

• $\text{OPT}(j, w) = \max \left\{ w_j + \text{OPT}(j - 1, w - w_j) \right\}$
Subset Sum Problem

• Start with a recursive algorithm:

```c
isSubsetSum(int set[], int n, int sum) {
    if (sum == 0) {
        return true;
    } //if
    if (n == 0 && sum != 0) {
        return false;
    } //if
    if (set[n-1] > sum) {
        return isSubsetSum(set, n-1, sum);
    } //if
    return isSubsetSum(set, n-1, sum) ||
        isSubsetSum(set, n-1, sum-set[n-1]);
} //isSubsetSum
```
Subset Sum Problem

• Convert to a bottom-up, array-based iterative algorithm:

```c
isSubsetSum(int set[], int n, int sum) {
    for (int i=0; i<sum; i++) {
        subsetsum[0][i] = 0;
    } //for
    for (int i=0; i<n; i++) {
        subsetsum[i][0] = 1;
    } //for
    for (int i=1; i<n; i++) {
        for (int j=1; j<sum; i++) {
            subsetsum[i][j] = max(
                subsetsum[i-1][j],
                subsetsum[i-1][j-set[j]]
            );
        } //for
    } //for
    return subsetsum[n][sum];
} //isSubsetSum
```
Subset Sum Problem

- Solve $v_1 = 1, v_2 = 2, v_3 = 3, v_4 = 4; L = 7$

\[
\begin{array}{cccccccc}
  & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
Subset Sum Problem

- What is our computational efficiency?
  - For the recursive method, $O(2^n)$
  - For the iterative method, $O(n \times L)$
Dynamic Programming

• Alternate approach: the Pruning Method.
  – At every decision, two options: (1) take the choice, or (2) don’t take the choice.
  – Build a tree based off of the two options.

• Pruning rules:
  1. If a node ever goes beyond the bounds of the problem, prune it.
  2. If there are 2+ nodes on the same level of the tree with the same value, prune all but one of them.
Subset Sum Problem

- Tree construction: Either take coin $v_i$ or don’t.
Subset Sum Problem

• Pruning Rules:
  1. If the coins at a node sum to >L, then prune the subtree at that node.
  2. If there are 2+ nodes on the same level with coins summing to the same value, prune all but one of those nodes.
Subset Sum Problem

• \( v_1 = 1, v_2 = 1, v_3 = 3 ; L = 3 \)
Subset Sum Problem

• So what does pruning get us?
  – There are at most \((n + 1)\) levels in the tree if we have \(n\) coins.
  – Each level can have at most \(L+1\) sums, because pruning guarantees unique sums capped at \(L\).

• So we fill an \((n + 1) \times (L + 1)\) matrix!
Subset Sum Problem

SubsetSum(int[] coins, int L) {
    A[0,0] = 1;
    for (int lvl = 0; lvl < n; lvl++) {
        for (int s = 0; s <= L; s++) {
            if (A[lvl, s]) {
                A[lvl+1, s] = 1;   //left child
                A[lvl+1, s+coin[lvl+1]] = 1 //r
            } //if
        } //for
    } //for
} //SubsetSum

A[lvl,s] = 1 if \exists subset of coins 1..lvl that sums to s
Subset Sum Problem

- Solve $v_1 = 1, v_2 = 2, v_3 = 3, v_4 = 4; L = 7$

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<th>lvl</th>
<th>s</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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</tbody>
</table>

- Dynamic Programming
Longest Common Subsequence

• **Input:**
  – String $A = A_1 A_2 \ldots A_m$
  – String $B = B_1 B_2 \ldots B_n$

• **Output:**
  – Length of the longest common subsequence (can skip letters).

• Example input: $A = xxyzzxyz$, $B = xyxxyyz$
• Example output: 5 (xyxyz)
Longest Common Subsequence

- Start with a recursive algorithm.

- \[ A \]
  - \[ \begin{align*} 0 & \quad m-1 \\ \text{or} & \\ 0 & \quad n-1 \end{align*} \]

- \[ B \]
  - \[ \begin{align*} 0 & \quad m-1 \\ \text{or} & \\ 0 & \quad n-1 \end{align*} \]
Longest Common Subsequence

• Start with a recursive algorithm.

```c
LCS(int m, int n) {
    if (m == 0 || n == 0) {
        return 0;
    } else if (A[m] == B[n]) {
        return LCS(m-1, n-1) + 1;
    } else {
        return max(LCS(m, n-1), LCS(m-1, n));
    } //if-else
} //LCS
```
Longest Common Subsequence

- Convert to iterative, bottom-up algorithm

```c
LCS(int m, int n) {
    for (int i = 0; i <= m; i++)
        LCS[i][0] = 0;
    for (int i = 0; i <= n; i++)
        LCS[0][i] = 0;
    for (int i = 1; i <= m; i++)
        for (int j = 1; j <= n; j++)
            if (A[i] == B[j])
                LCS[i][j] = LCS[i-1][j-1]+1;
            else
                LCS[i][j] = max(LCS[i][j-1], LCS[i-1][j]);
    return LCS[m][n];
} //LCS
```
Longest Common Subsequence

- Solve $LCS(xyxx, yxxxy)$

<table>
<thead>
<tr>
<th></th>
<th>y</th>
<th>x</th>
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<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
Longest Common Subsequence

• Can we recover that longest subsequence?

\[
\begin{array}{c|cc|cc|c}
& y & x & x & x & Y \\
0 & 0 & 0 & 0 & 0 & 0 \\
--- & --- & --- & --- & --- & --- \\
x & 1 & 0 & 0 & 1 & 1 \\
--- & --- & --- & --- & --- & --- \\
y & 2 & 0 & 1 & 1 & 1 \\
--- & --- & --- & --- & --- & --- \\
x & 3 & 0 & 1 & 2 & 2 \\
--- & --- & --- & --- & --- & --- \\
x & 4 & 0 & 1 & 2 & 3 \\
\end{array}
\]
Any Questions?